Ekman-Hartmann layer in a magnetohydrodynamic Taylor-Couette flow

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We study magnetic effects induced by rigidly rotating plates enclosing a cylindrical magnetohydrodynamic (MHD) Taylor-Couette flow at the finite aspect ratio $H/D=10$. The fluid confined between the cylinders is assumed to be liquid metal characterized by small magnetic Prandtl number, the cylinders are perfectly conducting, an axial magnetic field is imposed with Hartmann number $\mathrm{Ha}\approx 10$, and the rotation rates correspond to Reynolds numbers of order 10^2-10^3. We show that the end plates introduce, besides the well-known Ekman circulation, similar magnetic effects which arise for infinite, rotating plates, horizontally unbounded by any walls. In particular, there exists the Hartmann current, which penetrates the fluid, turns in the radial direction, and together with the applied magnetic field gives rise to a force. Consequently, the flow can be compared with a Taylor-Dean flow driven by an azimuthal pressure gradient. We analyze the stability of such flows and show that the currents induced by the plates can give rise to instability for the considered parameters. When designing a MHD Taylor-Couette experiment, special care must be taken concerning the vertical magnetic boundaries so that they do not significantly alter the rotational profile.

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INTRODUCTION

Motion of a fluid confined between two concentric, rotating cylinders is a classical problem in hydrodynamics and, if the fluid is conducting and an external magnetic field is applied, magnetohydrodynamics (MHD). Flow of this type, usually referred to as Taylor-Couette flow, was first studied by Couette [1] and later was the subject of a seminal work by Taylor [2], who experimentally confirmed the theoretical results of a linear stability analysis. In the field of MHD, important work was done by Velikhov [3] who showed that for a conducting fluid a weak magnetic field can play a destabilizing role and can lead to an instability which today is called the magnetorotational instability (MRI) [4].

When studying the Taylor-Couette system it is common to assume some simplifications, the small-gap approximation or large aspect ratio. In the former it is assumed that the gap between the cylinders $D=R_{\mathrm{out}}-R_{\mathrm{in}}$ is small compared to the radii, i.e., $D/R_{\mathrm{out}}\ll 1$; this allows the neglect of terms of order $1/R$, R being the distance from the center of rotation. When considering a large aspect ratio, one assumes that the height of the cylinders H is much larger than the gap width $\Gamma=H/D\gg 1$, which guarantees that a secondary flow due to the plates bounding the cylinders is insignificant and does not disturb the rotational profile of the fluid.

On the other hand, there is also plenty of work done for small aspect ratio $\Gamma\approx 1$, where the rigidly rotating end plates play a crucial role and simply introduce a new class of problems. When Γ becomes an important parameter, it is possible to observe a wide family of different states (including nonaxisymmetric ones or peculiar asymmetric patterns—anomalous modes) for the same parameters, so that the observed results depend on their path through the parameter space from an initial state. Therefore this system is an excellent subject for bifurcation theory [5–10].

In the present work, we focus on the case of a wide gap $R_{\mathrm{in}}/R_{\mathrm{out}}=1/2$ and $\Gamma=10$, which is an intermediate aspect ratio, between very short and long containers, yet in a purely hydrodynamical context the influence of the vertical boundaries is small, at least for Reynolds numbers of order $O(10^2-10^3)$. However, if the rotation rates are large enough, so that the corresponding Reynolds number is $O(10^5)$ and larger, the plates can easily dominate the flow in the entire container. This is due to the Taylor-Proudman theorem, from which it follows that in rapidly rotating systems the flow tends to align itself along the axis of rotation. For such rotations, it is necessary that Γ would have to be several thousand in order to obtain a rotational profile which is not profoundly altered by the end plates [11].

The results of a recent MRI experiment Potsdam-Rossendorf magnetic instability experiment (PROMISE) [12–14] as well as nonlinear simulations [15,16] indicate that, for a flow with relatively small Reynolds number $\approx 10^3$, and parameters resembling essentially MHD stable flow in the limit of infinitely long cylinders, there exist unexpected time-dependent fluctuations of the velocity field. These disturbances arise as an effect of the vertical boundary conditions; moreover, the simulations show that they are much stronger if the end plates bounding the cylinders are assumed to be perfectly conducting.

The plates induce a well-known hydrodynamical effect—the Ekman circulation, which is a result of unbalanced pressure gradients in the vicinity of the vertical no-slip boundary conditions. There the Ekman layer develops, in which the fluid velocity from the bulk of the container must match the velocity imposed by the end plates.

It seems that for MHD Taylor-Couette flow, magnetic effects, unlike the classical hydrodynamical Ekman layer, induced by the plates have been overlooked. In this paper, we argue that the rigidly rotating plates together with an imposed axial magnetic field give rise to a similar layer, which develops for an infinite, rotating plate serving as a boundary for the conducting fluid. One of the most important features

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of such flow is the existence of the Hartmann current (absent in the conventional Hartmann problem [17]), which leaves the boundary layer and then interacts with the magnetic field. In particular, this becomes important for conducting plates, which was the case for the PROMISE experiment, since one of the end plates was made from copper.

We discuss properties of Ekman-Hartmann layers for infinite, rotating plates and relate it to the end plates enclosing the cylinders in a Taylor-Couette setup. It is shown that, for the considered radial boundary conditions, the induced current turns eventually in the radial direction and, acting in concert with the imposed axial magnetic field, gives rise to a body force.

We demonstrate that magnetic effects induced by the end plates enclosing the cylinders can profoundly alter flow properties. In particular, the rotational profile can become significantly different from the expected parabolic Couette solution. Moreover, if the Hartmann current is strong enough, it is likely that the local Rayleigh criterion for stability will be violated and the flow becomes centrifugally unstable. In a MRI experiment it is crucial to rule out such instabilities, and special care concerning the vertical boundary conditions is needed in order to obtain the desired rotational profile.

PROBLEM FORMULATION

We consider two concentric cylinders with radii $R_{\text{in}}, R_{\text{out}}$ embedded in an external axial magnetic field. They rotate with angular velocities $\Omega_{\text{in}}, \Omega_{\text{out}}$, the radius ratio is $\hat{\eta} = R_{\text{in}}/R_{\text{out}}$, and the rotation ratio $\hat{\mu} = \Omega_{\text{out}}/\Omega_{\text{in}}$. Cylindrical coordinates (R, ϕ, z) with unit vectors $\hat{\mathbf{e}}_R, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_z$ are used. If the cylinders are unbounded, i.e., infinitely long or periodic, the rotational profile is

$$\Omega_0(R) = a + \frac{b}{R^2}, \tag{1}$$

with

$$a = \Omega_{\text{in}} \frac{\hat{\mu} - \hat{\eta}^2}{1 - \hat{\eta}^2}, \quad b = \frac{1 - \hat{\mu}}{1 - \hat{\eta}^2} R_{\text{in}}^2 \Omega_{\text{in}}, \tag{2}$$

and $u_R = u_z = 0$ everywhere. The flow is hydrodynamically stable if the Rayleigh criterion $d(R^2\Omega)^2/dR > 0$ is satisfied, i.e., for $\hat{\mu} > \hat{\eta}^2$. Consequently, for the considered radius ratio $\hat{\eta} = 1/2$, the flow is always stable if $\hat{\mu} > 0.25$. Here we consider only cases when $\hat{\mu} > 0.25$ so that hydrodynamical instabilities are ruled out.

Let us introduce the Reynolds number Re, which measures the rotation rates, and the Hartmann number Ha, which measures the strength of the externally applied magnetic field $\mathbf{B}_0 = B_0 \hat{\mathbf{e}}_z$,

$$\text{Ha} = B_0 \sqrt{\frac{D^2}{\mu_0 \rho \nu \eta}}, \quad \text{Re} = \frac{\Omega_{\text{in}} R_{\text{in}} D}{\nu}, \tag{3}$$

where ρ is the density, ν is the kinematic viscosity, η is the magnetic diffusivity, and μ_0 is the magnetic permeability. The fluid confined between the cylinders is assumed to be incompressible and it can be characterized by the magnetic Prandtl number $\text{Pm} = \nu/\eta$. For laboratory liquid metals, like gallium, Pm is very small—of order $10^{-(5\cdots 6)}$; therefore we concentrate on effects arising only when Pm is small.

The equations

Using D as the unit of length, ν/D as the unit of velocity, D^2/ν as the unit of time, B_0 as the unit of the axial magnetic field and assuming $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, we can write non-dimensional MHD equations for the problem of our interest, i.e.,

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nabla^2 \mathbf{u} + \frac{\text{Ha}^2}{\text{Pm}}[(\text{rot } \mathbf{b}) \times \mathbf{b} + (\text{rot } \mathbf{b}) \times \mathbf{B}_0/B_0], \tag{4a}$$

$$\partial_t \mathbf{b} = \frac{1}{\text{Pm}} \nabla^2 \mathbf{b} + \text{rot}(\mathbf{u} \times \mathbf{b}) + \text{rot}(\mathbf{u} \times \mathbf{B}_0/B_0), \tag{4b}$$

with div $\mathbf{u}$ = div $\mathbf{b} = 0$, where $\mathbf{u}$ and $\mathbf{b}$ are the velocity and the perturbed magnetic field, and p is the pressure.

For the velocity we apply no-slip boundary conditions at the cylinders and at the end plates as well. We assume that both the plates rotate rigidly with angular velocity Ω_{end}, which can be set to any value so that the plates can rotate independently of the cylinders.

Boundary conditions for the magnetic field are determined by magnetic properties of the cylinders and the plates. Here we consider only perfectly conducting radial boundaries, so that the transverse currents and perpendicular component of the magnetic field vanish; hence $R^{-1} b_\phi + \partial_R b_\phi = 0$ at $R = R_{\text{in}}/D, R = R_{\text{out}}/D$. We chose such boundaries since in the PROMISE experiment the cylinders were made of copper. The reason for choosing copper is that the critical Re and Ha numbers for the onset of the MRI are smaller by almost a factor of 2 with perfectly conducting boundaries than with insulating boundaries [18].

For the end plates, as for the walls, the electric field must be continuous and $b_z = 0$; then $b_\phi = \epsilon \partial_z b_\phi$ at $z = 0$ and $b_\phi = -\epsilon \partial_z b_\phi$ at $z = \Gamma$, where ϵ characterizes a thin layer of relative conductance of the fluid and the plates [19,20]. When $\epsilon \to 0$ we obtain conditions corresponding to insulating end plates, i.e., $b_\phi = \partial_z j_\phi = 0$ at $z = 0, z = \Gamma$, j_ϕ being the azimuthal current. For $\epsilon \to \infty$, we have the case describing perfectly conducting plates, $\partial_z b_\phi = j_\phi = 0$ at $z = 0, z = \Gamma$. We note that this thin-wall approximation is valid only when the magnetic field varies linearly within the plates and it does not necessarily resemble the situation in a real experiment.

The small-Pm limit

For laboratory liquids, the conductivity σ is small, so that the magnetic diffusivity $\eta = 1/\mu_0\sigma$ is very large (compared to the viscosity) and the corresponding magnetic Prandtl number Pm is small. Consequently, the time scale for magnetic diffusion is much shorter than other time scales. Therefore we consider the limit $\eta \to \infty$; however it must be supposed that Ha tends to a finite value. The perturbations $\mathbf{b}$ of the externally applied field induced by the motion of the fluid are

Pm times smaller than $\mathbf{B}_0$, although their effect on the Lorentz force cannot be neglected since $\mathrm{Ha}^2/\mathrm{Pm}[(\mathrm{rot}\,\mathbf{b})\times\mathbf{B}_0/B_0]$ is already of order Ha. Nevertheless, the interactions $(\mathrm{rot}\,\mathbf{b})\times\mathbf{b}$ are vanishingly small.

Similarly, in the induction equation we may apply a quasistatic approximation, so that the electromagnetic field proceeds along a sequence of steady-state solutions of the Maxwell equations to conditions described by $\mathbf{u}$, and therefore $\mathbf{b}$ in each moment adjusts instantaneously to the velocity $\mathbf{u}$. Hence, in the small-Prandtl-number limit $\mathrm{Pm}\to 0$, the system (4a) and (4b) can be written as

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nabla^2\mathbf{u} + \mathrm{Ha}^2(\nabla\times\mathbf{b})\times\mathbf{B}_0/B_0, \tag{5a}$$

$$\nabla^2\mathbf{b} = -\nabla\times(\mathbf{u}\times\mathbf{B}_0/B_0) \tag{5b}$$

with $\mathrm{div}\,\mathbf{u}=\mathrm{div}\,\mathbf{b}=0$ [21,22]. Equations (5a) and (5b) together with the discussed boundary conditions are solved with the finite-difference method using the stream function–vorticity formulation in the (R,z) plane. In this work we assume that the flow is axisymmetric. For more details on the numerical procedure, see [10,15].

THE EKMAN-HARTMANN LAYER IN MHD TAYLOR-COUETTE FLOW

At an interface between an incompressible fluid with low viscosity and a rapidly rotating rigid surface, an Ekman layer develops with thickness $d_{\mathrm{E}}\propto\sqrt{\nu/\Omega}$, where Ω is the rate of uniform rotation. Similarly, for a flow of conducting, incompressible fluid in the vicinity of a rigid nonrotating boundary and under the influence of an external magnetic field perpendicular to the surface, there exists a Hartman layer with thickness $d_{\mathrm{H}}\propto\mathrm{Ha}^{-1}$. When these two effects are combined, the Ekman-Hartmann layer develops [23]. It can be viewed either as a modification of the Ekman layer by introducing the conducting fluid and imposing the external magnetic field or as a modification of the Hartmann layer by adding the uniform rotation of the bounding surface. The resulting layer (in its steady form) assures a proper transition for the velocity and the magnetic field from the values inside the bulk of the fluid to the applied boundary conditions.

The linear analysis of the Ekman-Hartmann layer in its idealized case was presented in [24]. There, Gilman and Benton considered an infinite, insulating plate rotating with Ω_{plate} at $z=0$, and a conducting fluid filling the space $z>0$; the fluid far from the plate rotates with $\Omega_{\mathrm{fluid}}=\Omega_{\mathrm{plate}}(1+\epsilon')$, $\epsilon'\ll 1$. The most important conclusion of this work was that, in addition to the well-known Ekman suction or blowing of mass flux, there also exists an electric Hartmann current which has the same direction (or opposite when the external B_z is negative) as the velocity of the Ekman blowing (the fact that fluid is blown away or sucked toward the boundaries depends only on the sign of ϵ'). This current, which arises due to the vertical shears, leaves the Ekman-Hartmann layer and potentially influences the flow far away from the boundary.

For magnetized Taylor-Couette flow with finite aspect ratio, i.e., if the cylinders are covered with rigidly rotating end plates (insulating or conducting), the Ekman-Hartmann layer also develops. Naturally, the influence of the vertical walls introduces additional important effects, and direct quantitative comparison with the previous work is not possible. We must take into account that the fluid which was ejected due to the Ekman blowing mechanism must eventually get back due to the conservation of mass and finiteness of the container. Nevertheless, we will show that the rotating end plates induce a Hartmann current, which can change the global properties of the flow.

Let us introduce the parameter α which measures the overall importance of the magnetic field,

$$\alpha = \frac{d_{\mathrm{E}}}{\sqrt{2}d_{\mathrm{H}}} = \frac{\mathrm{Ha}}{\sqrt{2\,\mathrm{Re}\,\hat{\mu}}}, \tag{6}$$

where $d_{\mathrm{H}}=D\mathrm{Ha}^{-1}$ is the Hartmann depth. For the Ekman depth, as a measure of the uniform rotation we use Ω_{out}. The magnetic effects start to be significant when $\alpha\gtrsim 1$, in the limit $\alpha\to 0$ we have the classical Ekman layer, and for $\alpha\to\infty$ the classical Hartmann layer. We notice that, for slow rotation corresponding to Re of order $O(10^2-10^3)$ and Ha of order $O(10)$, $\alpha\approx 1$, and therefore we expect the magnetic fields to be important for many laboratory experiments.

Insulating end plates

First we consider a case when both the cylinders rotate with the same angular velocity $\Omega_{\mathrm{in}}=\Omega_{\mathrm{out}}=100$, i.e., $\hat{\mu}=1.0$, and the rotational profile (1) is flat. The aspect ratio is $\Gamma=10$ and the insulating plates rotate with angular velocity slightly different from that of the cylinders, $\Omega_{\mathrm{end}}=90$.

Figure 1 shows how the axial velocity u_z and the axial current j_z change with distance z from the plates, for different strength of the applied magnetic field. It can be seen that the axial velocity and the axial current decrease for stronger magnetic field. The explanation is as follows. The vertical shears in u_R and u_z produce currents which together with the axial field generate body forces acting against the shears. Since the radial flow must vanish at the boundaries as well as is vanishing far away from them, the effect is to reduce u_R and, due to mass conservation, u_z. Therefore the external axial magnetic field inhibits the Ekman blowing (which is completely suppressed when $\alpha\to\infty$) and makes the boundary layer thinner. The azimuthal flow u_ϕ, on the other hand, is forced to have different values at the boundaries and far away from them; thus the shear can be decreased only in the region close to the boundary.

These results are in a good agreement with the linear solution [24] for the case of the infinite, rotating plate and $\mathrm{Pm}\to 0$ (the agreement for other quantities like u_R and u_ϕ is pleasing as well). We notice that in the radially unbounded case u_z and j_z are independent of R, which cannot be true for the enclosed Taylor-Couette system. The values presented in Fig. 1 are computed for $R=R_{\mathrm{in}}+D/2$, in the middle of the gap, so that the influence of the rotating cylinders is as small as possible.

We point out that the induced axial current j_z (the Hartmann current) exists outside the boundary layer. This is not the case for nonrotating Hartmann boundaries. For un-

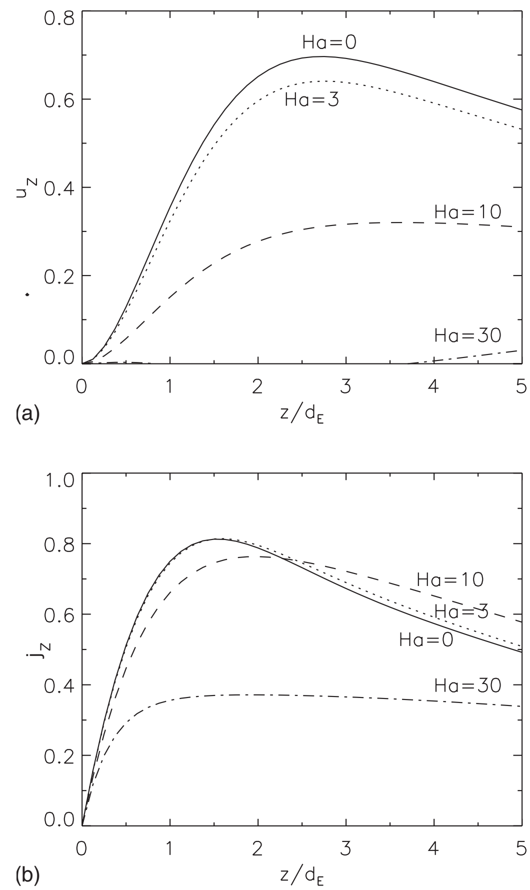

FIG. 1. Structure of the Ekman-Hartmann layer for the MHD Taylor-Couette flow with $\Omega_{\text{in}}=\Omega_{\text{out}}=100$, $\Gamma=10$, insulting end plates rotating with $\Omega_{\text{end}}=90$, and z the distance from one of the plates. Different values of the magnetic interaction parameters correspond to $\text{Ha}\rightarrow 0.0$ ($\alpha\rightarrow 0.0$), $\text{Ha}=3.0$ ($\alpha=0.2$), $\text{Ha}=10.0$ ($\alpha=0.7$), and $\text{Ha}=30.0$ ($\alpha=2.1$); $d_E=0.01D$ is the Ekman layer thickness.

bounded flow this current quickly converges to an asymptotic constant value, but for the case of flow between two plates, or for the enclosed cylinders, it cannot be true, and the currents induced by both end plates must eventually interact. When we consider a system symmetric in the z direction, i.e., when the two plates rotate in the same manner, the induced j_z have the same strength but opposite signs, and they eventually meet, turning in the radial direction (and consequently $j_z=0$ in the middle of the container for symmetric boundary conditions).

We have varied $0\leq\Omega_{\text{end}}\leq\Omega_{\text{out}}$ for constant $\Omega_{\text{in}}=\Omega_{\text{out}}=100$; similarly we considered $0\leq\Omega_{\text{in}}=\Omega_{\text{out}}\leq 100$ for $\Omega_{\text{end}}=100$ to get the values of the Ekman-Hartmann blowing and suction when the difference between cylinder and end plate rotation is large. The agreement with previous nonlinear calculations for the infinite plate is quite good [25]. The dependence of the induced mass flux and the current on the strength of the magnetic field as well as on the relative fluid–end plate rotation has the same character.

If the flow is vertically bounded by two plates, as for the Taylor-Couette system, three essentially different regions can be distinguished: the Ekman-Hartmann layer, a magnetic diffusion region, and a current-free region ([26,27]). In the magnetic diffusion region (MDR) the axial Hartmann current must be reduced to zero before it reaches the current-free region and, by continuity, it is turned to the radial direction. This radial perturbation current interacts with the axial magnetic field and results in an accelerating (for negative j_R and positive B_z) or decelerating (for positive j_R) electromagnetic body force.

The MDR arises since the Ekman-Hartmann layer itself is incapable of forcing the current to satisfy the exterior boundary conditions. It constantly grows in time and it quickly dominates the whole space between the plates. Moreover, when considering the small-Pm limit, the MDR instantly becomes spatially uniform and infinitely thick even for one bounding plane, and the current-free region does not exists at all [27].

Consequently, in our enclosed MHD Taylor-Couette system with $\text{Pm}\rightarrow 0$, we have a relatively thin Ekman-Hartmann layer close to the plates, whereas the fluid in the major part of the container forms the MDR in which the axial Hartmann current changes to a radial one. We underline here that this is true for perfectly conducting walls, since such radial boundary conditions assure us that the current can penetrate the cylinders. The situation would be rather different with insulating radial boundaries.

Conducting end plates

For highly conducting plates, the induced current drawn into or from the plates is much stronger than the current induced in the layer for insulating boundaries. The Ekman-Hartmann layer itself is nearly unaffected by conductivity of the plates, as are the velocities and the currents within this layer. However, due to the constant magnetic field perturbation there exists an additional electric current of order 2Φ which is induced by the conducting boundaries, $\Phi=\epsilon\sqrt{\Omega_{\text{plate}}/\nu}$, and ϵ characterizes the relative conductance of the fluid and the thin plates [19]. Moreover, in the MDR this current increases the fluid velocity by a factor of $2\alpha^2\Phi$.

Figure 2 shows how the radial current j_R in the middle of the container changes with the conductivity of the end plates. The difference between the perfect insulator and the perfect conductor is almost one order of magnitude even for such slow rotation. We find that for an MRI experiment it is *crucial to use insulating plates* in order to minimize this undesirable current.

THE INFLUENCE OF THE HARTMANN CURRENT

We notice that the force due to the radial current and the axial magnetic field, $\text{Ha}^2 j_R\hat{\mathbf{e}}_R\times\mathbf{B}_0/B_0$, enters the momentum equation (5a) for the u_ϕ component. Formally, the force is equivalent to applying an azimuthal pressure gradient $\partial_\phi p\neq 0$. A flow between rotating cylinders with nonzero $\partial_\phi p$ is usually referred to as Taylor-Dean flow [28]. Its rotational profile Ω_{D} is a superposition of the circular Couette profile (1) and the steady flow

$$\Omega_{\text{D}}=\Omega_0+e(c+d/R^2+\ln R), \tag{7}$$

with

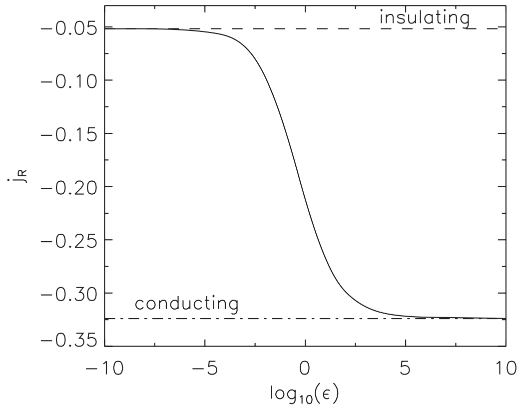

FIG. 2. Radial current j_R $(R=R_{\rm in}/D+1/2, z=\Gamma/2)$ in the middle of the gap for MHD Taylor-Couette flow with Ha $=10$, $\Omega_{\rm out}=\Omega_{\rm in}=200$, $\Omega_{\rm end}=202$, $\Gamma=10$. The upper line represents insulating plates, the bottom line perfectly conducting ones, and in between is the intermediate case for different values of the relative conductance.

$$c=\frac{R_{\rm in}^2\ln(R_{\rm in})-R_{\rm out}^2\ln(R_{\rm out})}{R_{\rm out}^2-R_{\rm in}^2}, \tag{8}$$

$$d=\frac{R_{\rm in}^2R_{\rm out}^2\ln(R_{\rm out}/R_{\rm in})}{R_{\rm out}^2-R_{\rm in}^2}, \tag{9}$$

$$e=\frac{1}{\rho\nu}\partial_\phi p. \tag{10}$$

The pressure gradient can be realized by an external pumping mechanism or, as in the case discussed, by the Lorentz force resulting from the induced current and the axial magnetic field.

Let us introduce a parameter β describing Taylor-Dean flows, the ratio of the average pumping velocity to the rotation velocity,

$$\beta=\frac{6V_m}{\Omega_{\rm in}R_{\rm in}}, \tag{11}$$

where V_m is the average pumping velocity,

$$V_m=\frac{1}{D}\int_{R_{\rm in}}^{R_{\rm out}}[e(c+d/R^2+\ln R)]dR$$
$$=-\partial_\phi p\frac{R_{\rm out}}{2\rho\nu}\frac{(1-\hat\eta^2)^2-4\hat\eta^2(\ln\hat\eta)^2}{4(1-\hat\eta)(1-\hat\eta^2)} \tag{12}$$

[29]. The basic question arises whether the resulting pumping due to the radial current and the axial field can bring the flow into an unstable regime.

Hartmann current generated by the end plates

The structure of the Ekman-Hartmann layer changes with parameters such as rotation rate or strength of the magnetic field. Here, however, we will concentrate on the flow in the bulk of the container so that only currents and velocities which leave the layer are important. We analyze hydrodynamically stable flow with $\hat\mu=0.27$ at the aspect ratio $\Gamma=10$ with rigidly rotating end plates.

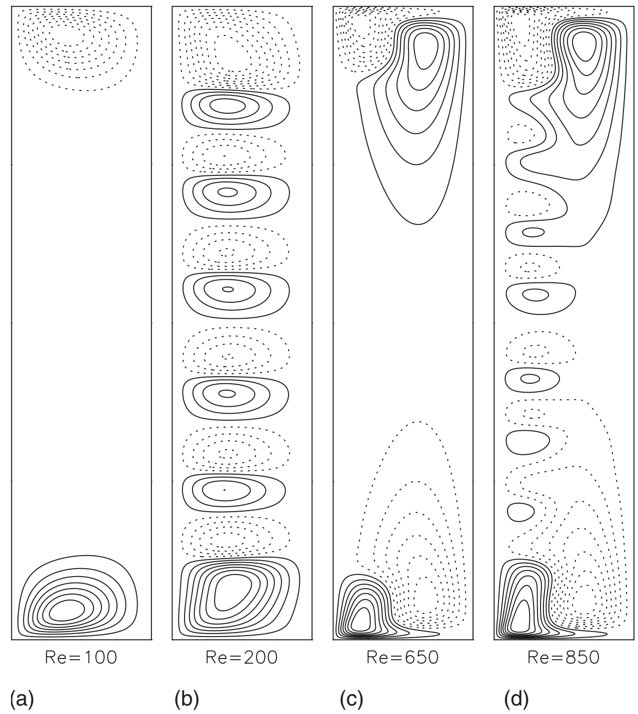

FIG. 3. Contour lines of stream function for different flow parameters: the left edge of each panel denotes the inner cylinder, the right edge the outer one, and solid lines correspond to clockwise fluid rotation. The end plates are attached to the outer cylinder, $\Omega_{\rm end}=\Omega_{\rm out}$, $\hat\mu=0.27, \Gamma=10$. Cases a and b are for conducting plates and Ha=3, c and d for insulating ones and Ha=10.

End plates rotating with $\Omega_{\rm out}$ and $\Omega_{\rm in}$

First we consider cylinders covered with rigid, perfectly conducting plates rotating with angular velocity equal to that of the outer cylinder, $\Omega_{\rm end}=\Omega_{\rm out}$. We choose conducting lids so that the induced current is much stronger and its influence on the flow is more evident.

When the plates rotate with $\Omega_{\rm end}=\Omega_{\rm out}$, the Ekman circulation is clockwise and the corresponding Hartman current has the positive sign, i.e., close to the inner cylinder it leaves the Ekman-Hartmann layer with $j_z>0$; consequently the radial current also has positive sign. Figure 3, cases a and b, displays a flow with conducting plates and a weak axial magnetic field applied, Ha=3, for two different Reynolds numbers. The rotation ratio is $\hat\mu=0.27$, so that the Couette flow is hydrodynamically *stable*; however we notice that when Re is large enough the flow changes significantly and Taylor vortices can be observed.

This phenomenon can be explained as follows: For a constant Ha, increase of the rotation rate leads to a stronger Hartmann current drawn into the flow; therefore the corresponding pumping β due to $j_R\hat{\mathbf{e}}_R\times\mathbf{B}_0$ increases and for a certain Re it reaches a critical value β_c, so that the instability develops.

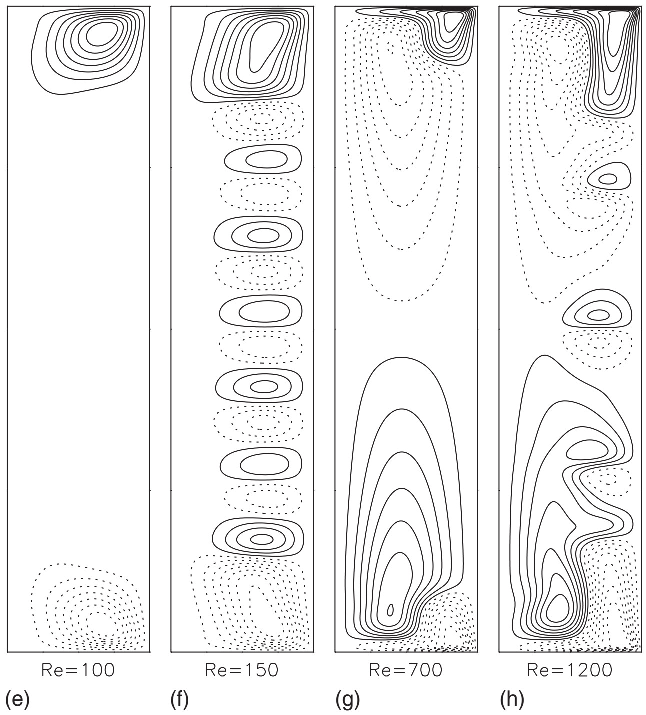

FIG. 4. As in Fig. 3, but end plates are attached to the inner cylinder, $\Omega_{\rm end}=\Omega_{\rm in}$. Cases e and f are for conducting plates and Ha=3, g and h for insulating ones and Ha=10. Note that for the insulating end plates Re is an order of magnitude larger.

If the perfectly conducting ends are replaced with insulating ones, the induced current is much weaker. When the imposed magnetic field has strength such that Ha=3, the pumping is too small to make the flow unstable, regardless of the Reynolds number. However, when the magnetic field is stronger, Ha=10, for sufficiently high rotation rates the vortices can also be seen, Fig. 3, cases c and d.

It is known that a stronger axial magnetic field has a stabilizing effect even on a hydrodynamically unstable flow (e.g., [30]). Besides that, the Hartmann current increases with the amplitude of the magnetic field only until a certain point is reached. When the magnetic interaction parameter becomes $\alpha\approx 2.5$, increasing Ha does not further increase the Hartmann current [24]. For these reasons it is clear that, when the imposed magnetic field is strong enough, the instability described above will not occur. Indeed, it has been checked that for conducting plates, Re=200, and a magnetic field with Ha=20, there are no Taylor vortices, although the rotational profile is significantly changed when compared to the nonmagnetic situation.

If rigidly conducting end plates are attached to the inner cylinder, so that $\Omega_{\rm end}=\Omega_{\rm in}$, the Ekman circulation is counterclockwise (Ekman suction) and the corresponding Hartmann current has a negative sign, so that the parameter β is positive. From Fig. 4, cases e and f, we see that, analogously to the cases c and d, if the rotation is sufficiently fast the resulting β reaches a critical value and the flow becomes dominated by the vortices.

Similarly when insulating plates are used, the axial magnetic field with Ha=3 is too weak to generate sufficiently large β. When a stronger field is applied, Ha=10, it is possible to observe the instability (cases g and h).

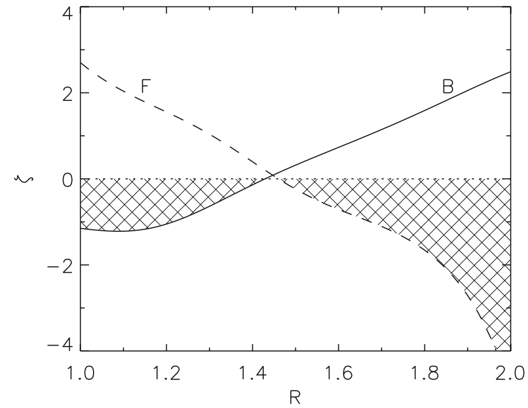

FIG. 5. $\zeta(R,z=\Gamma/2)$ calculated for MHD Taylor-Couette flow with conducting end plates attached to the outer cylinder (case b, Re=200) and the inner one (case f, Re=150). The Rayleigh criterion yields $\zeta>0$ for stability.

The rotational profile

As mentioned above, if the induced radial current has the same sign as the axial magnetic field, the azimuthal velocity of the fluid is decelerated; if the signs are opposite the flow is accelerated. The discussed instability is a centrifugal one and is simply due to change in the rotational profile of the fluid. Let us use a Rayleigh discriminant for stability, $\zeta=\partial_R(R^2\Omega)/(R\Omega)$; the flow is stable if $\zeta>0$. Figure 5 shows the radial dependence of ζ in the middle of the gap ($z=\Gamma/2$) for the two cases labeled as b and f.

We notice that the vortices concentrate in the region where ζ is negative, i.e., where the Rayleigh criterion is not satisfied. This instability has essentially local character, so it is not possible to define any specific critical Reynolds number whose crossing would lead to some exponential grow in the whole container. For conducting plates and $\Omega_{\rm end}=\Omega_{\rm out}$, there exists Re between 100 (case a) and 200 (case b) for which only a part of the container would be filled with the vortices.

Linear stability of current-induced MHD Taylor-Dean flow

In order to predict the onset of the instability discussed above, we analyze the global stability of MHD Taylor-Dean flow for our parameters. For the nonlinear simulations we can estimate the pumping due to the azimuthal pressure gradient just by setting $(\nabla p)_\phi$ to $\mathrm{Ha}^2 j_R$ see [Eq. (5a)]. Generally $(\nabla p)_\phi$ and j_R change with radius as R^{-1}. However, due to the presence of the plates, for j_R this is true only far from the vertical boundaries and here the value of j_R is taken at $R=R_{\rm in}/D$, $z=\Gamma/2$ (note that for our perfectly conducting boundaries the current penetrates the cylinders and for a steady state it is largest at $R=R_{\rm in}/D$). In this way we obtain the parameter β associated with the enclosed MHD Taylor-Couette flow for the given boundary conditions a–h, and then it can be compared with the critical value β_c obtained from the linear stability analysis.

Consider now the axisymmetric MHD Taylor-Dean flow for infinitely long cylinders governed by Eqs. (4a) and (4b). It admits the basic solution $u_\phi = R\Omega_D$ with $u_R = u_z = b_R = b_\phi = 0$ and the imposed axial magnetic field B_0. The perturbed state is $u'_R, R\Omega_D + u'_\phi, u'_z, b'_R, b'_\phi, B_0 + b_z$.

After developing disturbances into normal modes we seek solutions of the linearized MHD equations in a form similar to that in [30,31]. An appropriate set of ten boundary conditions is needed in order to solve the system; these are the no-slip boundary conditions for the velocity $u'_R = u'_\phi = u'_z = 0$ and perfectly conducting conditions for the magnetic field $\partial_R b'_\phi + b'_\phi / R = b'_R = 0$ at both cylinders. We will consider only stationary marginally stable modes.

The homogeneous set of equations together with the boundary conditions for the walls determine an eigenvalue problem of the form $L(\hat{\mu}, \hat{\eta}, k, m, \mathrm{Pm}, \mathrm{Re}, \mathrm{Ha}, \beta) = 0$. The variables are approximated with the finite-difference method on a grid typically with 200 points. The numerical code used to solve the problem is identical to that used in [30].

For the current axisymmetric study (see, however, [34]), we set parameters $m = 0$, $\hat{\eta} = 0.5$, $\hat{\mu} = 0.27$, $\mathrm{Pm} = 10^{-6}$; then for given Ha and Re we look for minimal value of $|\beta|$ leading to the instability (the value for which the determinant L is zero). Since β is directly proportional to the azimuthal pressure gradient, and therefore to the radial current, the resulting critical β_c determines the minimum value of the radial current for which the Taylor-Dean flow becomes unstable.

Figure 6 shows marginal stability lines for the MHD Taylor-Dean flow for different values of the imposed axial magnetic field, for both positive and negative values of β. We notice that much larger values of $|\beta|$ are needed for stronger axial magnetic fields since the field plays a stabilizing role.

The labels A–H refer to the MHD Taylor-Coutte flows presented in the previous section, e.g., a refers to the flow with $\mathrm{Re} = 100$, $\mathrm{Ha} = 3$, $\hat{\mu} = 0.27$ with perfectly conducting end plates attached to the outer cylinder. The induced current j_R is such that the corresponding β due to the Lorentz force denotes a stable flow. If the Reynolds number is increased, the critical value β_c (for Ha=3) is reached and an instability develops—label b.

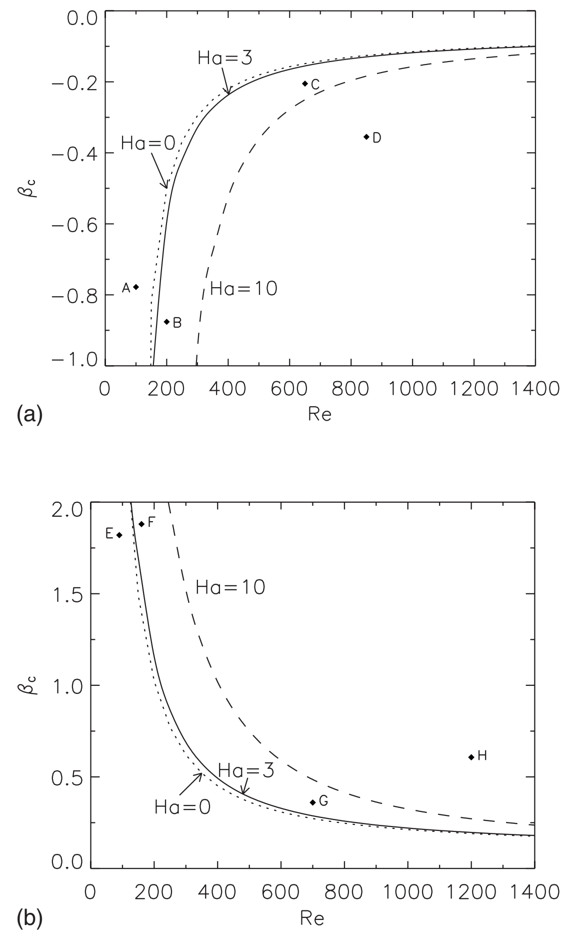

FIG. 6. Critical values of the pumping to rotation ratio β_c for MHD Taylor-Dean flow with $\hat{\mu} = 0.27$, for different Reynolds numbers Re and strength of the magnetic field Ha. The case for negative (positive) β corresponds to the azimuthal pressure gradient due to positive (negative) radial currents interacting with the axial field. On the left panel the unstable region lies below the line, on the right one above the lines. The letters a–h represent states displayed in Figs. 3 and 4.

SUMMARY

Gilman and Benton [24] have shown with a linear theory that, in the vicinity of a rotating plane which serves as a border for a rotating conducing fluid, the Ekman-Hartmann layer develops if $\Omega_{\mathrm{plate}} \neq \Omega_{\mathrm{fluid}}$ and an axial magnetic field is applied. The most important feature of Ekman-Hartmann layers is their ability to induce both mass fluxes and electric currents in the region outside the boundary layer. If $\Omega_{\mathrm{plate}} < \Omega_{\mathrm{fluid}}$ these fluxes are directed out of the layer (blowing); when $\Omega_{\mathrm{plate}} > \Omega_{\mathrm{fluid}}$ they are toward the layer (suction). For the conducting plates the fluxes are much stronger since additional currents are drawn from or into the plates.

Outside the Ekman-Hartmann layer is the magnetic diffusion region, in which the electric current has only radial components. The current, together with the axial magnetic field, produces an electromagnetic body force acting on the fluid.

We have shown in this paper that similar effects arise for the MHD Taylor-Couette flow when the rotating cylinders are bounded by two rigidly rotating end plates. Near the plates, the Ekman-Hartmann layer forms and, consequently, there exists a Hartmann current which penetrates the bulk of the fluid. In the presence of an axial magnetic field, this problem can be compared with the Taylor-Dean flow—a flow between (possibly rotating) cylinders which is additionally driven by an azimuthal pressure gradient.

We find that under certain conditions the resulting flow becomes unstable, Taylor vortices can be observed, and the rotational profile is significantly different from the standard Couette solution Ω_0. The instability has essentially a centrifugal character as the Rayleigh criterion is locally violated. This is an undesirable effect from the point of view of a MRI experiment. In such an experiment it is necessary to obtain a state resembling Ω_0 in the major part of the container, for parameters characterizing stable MHD flows. It is necessary to take into account the magnetic effects induced by the plates so that the MRI can be clearly identified rather than any other instability.

The fluxes induced in the Ekman-Hartmann layer are a direct consequence of a shear close to the boundaries. Exemplary methods of reducing the shear have been proposed in [16]. For rotation rates characterized by Re of order $O(10^3)$ all the effects can be significantly reduced by allowing the end plates to rotate independently of the cylinders [32]. Since for $\Omega_{end}=\Omega_{in}$ there is the Ekman suction and for $\Omega_{end}=\Omega_{out}$ the Ekman blowing, there exists $\Omega_{out}<\Omega_{end}<\Omega_{in}$ for which the generated mass and charge fluxes are minimal. Alternatively, one can divide the plates into independently rotating rings [33].

[1] M. M. Couette, C. R. Hebd. Seances Acad. Sci. **107**, 388 (1888).
[2] G. I. Taylor, Philos. Trans. R. Soc. London, Ser. A **223**, 289 (1923).
[3] E. P. Velikhov, Sov. Phys. JETP **36**, 1398 (1959).
[4] H. Ji, J. Goodman, and A. Kageyama, Mon. Not. R. Astron. Soc. **325**, L1 (2001).
[5] G. Pfister, H. Schmidt, K. A. Cliffe, and T. Mullin, J. Fluid Mech. **191**, 1 (1988).
[6] J. M. Lopez and F. Marques, Phys. Rev. E **68**, 036302 (2003).
[7] T. Mullin, Y. Toya, and S. J. Tavener, Phys. Fluids **14**, 2778 (2002).
[8] H. Furukawa, T. Watanabe, Y. Toya, and I. Nakamura, Phys. Rev. E **65**, 036306 (2002).
[9] A. Kageyama, H. Ji, J. Goodman, F. Chen, and E. Shoshan, J. Phys. Soc. Jpn. **73**, 2424 (2004).
[10] A. J. Youd and C. F. Barenghi, J. Fluid Mech. **550**, 27 (2006).
[11] R. Hollerbach and A. Fournier, in *MHD Couette Flows: Experiments and Models*, edited by R. Rosener, G. Rüdiger, and A. Bonanno, AIP Conf. Proc. No.733 (AIP, Melville, NY, 2004), pp. 114–121.
[12] F. Stefani, T. Gundrum, G. Gerbeth, G. Rüdiger, M. Schultz, J. Szklarski, and R. Hollerbach, Phys. Rev. Lett. **97**, 184502 (2006).
[13] G. Rüdiger, R. Hollerbach, F. Stefani, T. Gundrum, G. Gerbeth, and R. Rosner, Astrophys. J. Lett. **649**, L145 (2006).
[14] F. Stefani, T. Gundrum, G. Gerbeth, G. Rüdiger, J. Szklarski, and R. Hollerbach, New J. Phys. **9**, 295 (2007).
[15] J. Szklarski and G. Rüdiger, Astron. Nachr. **327**, 844 (2006).
[16] J. Szklarski, Astron. Nachr. **328**, 499 (2007).
[17] D. S. Krasnov, E. Zienicke, O. Zikanov, T. Boeck, and A. Thess, J. Fluid Mech. **504**, 183 (2004).
[18] G. Rüdiger, R. Hollerbach, M. Schultz, and D. A. Shalybkov, Astron. Nachr. **326**, 409 (2005).
[19] D. E. Loper, J. Fluid Mech. **13**, 2999 (1970).
[20] R. Hollerbach, E. Canet, and A. Fournier, Eur. J. Mech. B/Fluids **26**, 729 (2007).
[21] P. H. Roberts, *An Introduction to Magnetohydrodynamics* (Longmans, Greeen and Co., London, 1967).
[22] O. Zikanov and A. Thess, J. Fluid Mech. **358**, 299 (1998).
[23] D. J. Acheson and R. Hide, Rep. Prog. Phys. **36**, 159 (1973).
[24] P. A. Gilman and E. R. Benton, Phys. Fluids **11**, 2397 (1968).
[25] E. R. Benton and J. H. S. Chow, Phys. Fluids **15**, 2174 (1972).
[26] D. E. Loper and E. R. Benton, J. Fluid Mech. **43**, 785 (1970).
[27] E. R. Benton and D. E. Loper, J. Fluid Mech. **39**, 561 (1969).
[28] S. Chandrasekhar, Proc. Natl. Acad. Sci. U.S.A. **46**, 253 (1960).
[29] F. Chen, Phys. Rev. E **48**, 1036 (1993).
[30] G. Rüdiger, M. Schultz, and D. Shalybkov, Phys. Rev. E **67**, 046312 (2003).
[31] G. Rüdiger and D. Shalybkov, Phys. Rev. E **69**, 016303 (2004).
[32] J. Abshagen, K. A. Cliffe, J. Langenberg, T. Mullin, G. Pfister, and S. J. Tavener, Theor. Comput. Fluid Dyn. **18**, 129 (2004).
[33] H. Ji, M. Burin, E. Schartman, and J. Goodman, Nature (London) **444**, 343 (2006).
[34] It should be mentioned that, for a Taylor-Dean flow, it is fairly easy to excite oscillatory, nonaxisymmetric modes (see, e.g., [29]).